Wasserstein-1 distance between SDEs driven by Brownian motion and stable process

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Content

Based on a joint work with R. Schilling (Dresden) and L. Xu (Macau): arXiv: 2302.03372

1. **Background and motivation**
2. **Main result**
3. **Sketch of proof**

Background

Consider

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}L_t, \quad X_0 = x,$$
$$\mathrm{d}Y_t = b(Y_t)\,\mathrm{d}t + \mathrm{d}B_t, \quad Y_0 = x,$$

where L_t is an α-stable process and B_t is a B.M. in $\mathbb{R}^d$.

- $\mathbb{E}\, e^{i\xi L_t} = e^{-t|\xi|^\alpha} \xrightarrow{\alpha\uparrow 2} e^{-t|\xi|^2} = \mathbb{E}\, e^{i\xi B_t}$
- $L_t \xrightarrow{\alpha\uparrow 2} B_t$
- Natural question: $X_t \xrightarrow{\alpha\uparrow 2} Y_t$?

  Liu (JMAA 2022, SPA 2022, JDE 2022)
- Further (informal) question: $X_\infty \xrightarrow{\alpha\uparrow 2} Y_\infty$?

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- Ergodicity: $\mathscr{L}_{X_t} \to \mu_\alpha$ and $\mathscr{L}_{Y_t} \to \mu_2$, as $t \to \infty$.
- Our question: $\mu_\alpha \xrightarrow{\alpha \uparrow 2} \mu_2$?
- Convergence rate?

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Main result

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Assumption: $\|\nabla b\|_\infty < \infty$, $\|\nabla^2 b\|_\infty < \infty$, $\|\nabla^3 b\|_\infty < \infty$, and

$$\limsup_{|x-y|\to\infty} \frac{\langle x-y, b(x)-b(y)\rangle}{|x-y|^2} < 0.$$

Typical Exam.: $b(x) = -x+$ 'small pertubation'.

Theorem (D.-Schilling-Xu, 2023+)

For any $x, y \in \mathbb{R}^d$ and $t > 0$,

$$W_1\left(\mathscr{L}_{X_t^x}, \mathscr{L}_{Y_t^y}\right) \le C_1 \mathrm{e}^{-C_2 t}|x-y| + C(2-\alpha).$$

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Corollary (D.-Schilling-Xu, 2023+)

$$W_1(\mu_\alpha, \mu_2) \le C(2-\alpha).$$

Proof:

$$W_1(\mu_\alpha, \mu_2) \le W_1\left(\mu_\alpha, \mathscr{L}_{X_t^x}\right) + W_1\left(\mathscr{L}_{X_t^x}, \mathscr{L}_{Y_t^y}\right) + W_1\left(\mathscr{L}_{Y_t^y}, \mu_2\right).$$

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Optimal rate: $2-\alpha$? O-U case

$$\mathrm{d}X_t = -X_t\,\mathrm{d}t + \mathrm{d}L_t, \quad \mu_\alpha = \mathscr{L}_{\alpha^{-1/\alpha}L_1},$$
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$$\begin{aligned}
W_1(\mu_\alpha,\mu_2) &= \inf_{\Pi\in\mathscr{C}(\mu_\alpha,\mu_2)} \iint |x-y|\,\Pi(\mathrm{d}x,\mathrm{d}y) \\
&\geq \inf_{\Pi\in\mathscr{C}(\mu_\alpha,\mu_2)} \left| \iint |x|\,\Pi(\mathrm{d}x,\mathrm{d}y) - \iint |y|\,\Pi(\mathrm{d}x,\mathrm{d}y) \right| \\
&= \left| \int |x|\,\mu_\alpha(\mathrm{d}x) - \int |y|\,\mu_2(\mathrm{d}y) \right| \\
&= \left| \mathbb{E}\left|\alpha^{-1/\alpha}L_1\right| - \mathbb{E}\left|2^{-1/2}B_1\right| \right| \\
&\asymp (2-\alpha).
\end{aligned}$$

Remark: The rate $2-\alpha$ is sharp for the O–U case.

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Sketch of proof

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}L_t, \qquad P_t,\ \mathscr{A}^P,$$
$$\mathrm{d}Y_t = b(Y_t)\,\mathrm{d}t + \mathrm{d}B_t, \qquad Q_t,\ \mathscr{A}^Q.$$

Proof: It suffices to bound $|\mathscr{A}^P - \mathscr{A}^Q|$ since (Duhamel formula)

$$\begin{aligned} W_1\left(\mathscr{L}_{X_t^x}, \mathscr{L}_{Y_t^y}\right) &= \sup_{h\in \mathrm{Lip}(1)} |P_t h(x) - Q_t h(x)| \\ &= \sup_{h\in \mathrm{Lip}(1)} \left| \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} Q_{t-s} P_s h(x)\,\mathrm{d}s \right| \\ &= \sup_{h\in \mathrm{Lip}(1)} \left| \int_0^t Q_{t-s}(\mathscr{A}^P - \mathscr{A}^Q) P_s h(x)\,\mathrm{d}s \right|. \end{aligned}$$

$$\mathscr{A}^P f = \langle \nabla f, b\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left[f(\cdot + z) - f(\cdot) - \langle \nabla f(\cdot), z\rangle \mathbb{1}_{\{|z|\le 1\}}\right] \frac{C_{d,\alpha}}{|z|^{d+\alpha}}\,\mathrm{d}z,$$
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In order to bound $|(\mathscr{A}^P - \mathscr{A}^Q)P_s h|$, we need the following gradient estimate.

Lemma

For any $h \in \mathrm{Lip}(1)$ and $t \in (0,1]$,

$$\|\nabla P_t h\|_\infty \le C,$$
$$\|\nabla^2 P_t h\|_\infty \le C t^{-1/\alpha},$$
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Remark: For $h \in \mathscr{B}_b(\mathbb{R}^d)$ and $t \in (0,1]$,

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See e.g. Zhang, SPA 2013.

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Technique: Time-change ($L_t = B_{S_t}$) and Malliavin calculus.

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Summary

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Our result: $W_1(\mu_\alpha, \mu_2) \le C(2-\alpha)$.

Question: 1) Other distance $d(\mu_\alpha, \mu_2)$?

2) More general coefficients?

3) ......

Summary

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Assumption: $\|\nabla b\|_\infty < \infty$, $\|\nabla^2 b\|_\infty < \infty$, $\|\nabla^3 b\|_\infty < \infty$, and

$$\limsup_{|x-y|\to\infty} \frac{\langle x-y, b(x)-b(y)\rangle}{|x-y|^2} < 0.$$

Our result: $W_1(\mu_\alpha, \mu_2) \le C(2-\alpha)$.

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The end

Thanks for Your Attention!